Iitaka dimension

X an irred. projective variety, L a line bundle on X.

The semi-group of L $= \mathbb{N}(L) = \{ m \geq 0 \mid H^0(X, L^{\otimes m}) \neq 0 \}$
(where $L^{\otimes 0} = \mathcal{O}$)

Note: this is in fact a semi-group since if $m, n \in \mathbb{N}(L)$,
then the product of two nonzero sections will have nonzero image
in $H^0(L^m) \otimes H^0(L^n) \longrightarrow H^0(L^{m+n})$

If $\mathbb{N}(L) \neq 0$, the exponent of $L = e(L) =$ g.c.d. of elements of L.
All sufficiently large multiples of $e(L)$ are in $\mathbb{N}(L)$.

Ex: Let C be hyperelliptic, and choose a g^1_2 so that C is
ramified at P and Q.

Consider $L = \mathcal{O}(P - Q)$. Then $m \in \mathbb{N}(L) \Leftrightarrow L^{\otimes m} \cong \mathcal{O}_C \Leftrightarrow mP \equiv mQ$.
Thus, $\mathbb{N}(L) = 2\mathbb{N}$.

Exercise: Does a similar argument work for trigonal curves? 4-gonal
curves?

If $m \in \mathbb{N}(L)$, some line bundle L on X, there's a rational map

$$\varphi_m = \varphi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P} H^0(L^{\otimes m})$$

Let $Y_m = \overline{\varphi(X)} \subseteq \mathbb{P}H^0(L^{\otimes m})$.

Rmk: If L is ample, then for $m \gg 0$, $Y_m \cong X$.

Motivation: Understand Y_m as $m \to \infty$.

Def: If X is normal and $\mathbb{N}(L) \neq 0$, then the Iitaka dimension of L (or the L-dimension) is

$$\kappa(L) = \max_{m \in \mathbb{N}(L)} \{\dim Y_m\}$$

If $H^0(L^{\otimes m}) = 0$ for all $m > 0$, then set $\kappa(L) = -\infty$.

If X is non-normal take its normalization $\nu : X' \to X$ and set $\kappa(X, L) = \kappa(X', \nu^* L)$.

So $\kappa(L) = -\infty$ or $0 \leq \kappa(L) \leq \dim X$.

Ex: In the hyperelliptic curve example, $\kappa(\mathcal{O}_C(P-Q)) = 0$.

Ex: a.) X = blowup of $\mathbb{P}^2$ at a point. Then $L = \mathcal{O}_X(H)$ has Iitaka dim 2, but $L|_E \cong \mathcal{O}_E$, which has Iitaka dim 0.

b.) $X = \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}(-1) \boxtimes \mathcal{O}(1)$. Then $\kappa(L) = 0$, but $\kappa(L|_{\{P\} \times \mathbb{P}^1}) = 1$. If $L = \mathcal{O} \boxtimes \mathcal{O}(1)$, then $\kappa(L) = \kappa(L|_{\{P\} \times \mathbb{P}^1}) = 1$

i.e. Iitaka dim can increase or decrease under restriction.

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Prop: Let X be normal, and L a l.b. on X, with $\kappa := \kappa(L) \in \mathbb{N}$.
Then there is some $M > 0$ s.t. $\dim Y_m = \kappa$ $\forall$ $m \in \mathbb{N}(L)$ s.t. $m \geq M$.

Pf: Let e = exponent of L. Replace L w/ $L^{\otimes e}$.

Assume $\dim Y_k = \kappa$, and $j \in \mathbb{N}(L)$ $\forall$ $j \geq J$. Set $M = k + J$

Let $j \geq J$, and $0 \neq s \in H^0(L^j)$ Consider map

$$H^0(L^k) \xrightarrow{\cdot s} H^0(L^{k+j}).$$

This is an injection, so $\mathbb{P}(H^0(L^{k+j})) \dashrightarrow H^0(L^k)$ projection.

$$\begin{tikzcd} \mathbb{P}(H^0(L^{k+j})) \arrow[r, dashed] & H^0(L^k) \\ X \arrow[u, dashed] \arrow[ur, dashed] & \end{tikzcd}$$

and so $\kappa \geq \dim Y_{k+j} \geq \dim Y_k = \kappa$. $\square$

Def: X sm. proj. variety, then $\kappa(X) = \kappa(X, K_X)$ is the Kodaira dimension of X. (If X is singular, $\kappa(X) := \kappa(\tilde{X})$, $\tilde{X}$ a sm. model)

Def: X irr. proj. variety, D a Cartier divisor w/ $\kappa(D) \geq 0$
The <u>stable base locus</u> of D is

$$\mathbb{B}(D) = \bigcap_{m \geq 1} Bs(|mD|) \leftarrow \text{set-theoretic}$$

Prop: a.) $\mathbb{B}(D)$ = unique minimal elt of $\{Bs(|mD|)\}_{m \geq 1}$

b.) $\exists\, m_0 \in \mathbb{N}$ s.t. $\mathbb{B}(D) = Bs(|km_0 D|)\ \ \forall\ k \gg 0$.

<u>Pf</u>: Take m, ℓ. Then

$$H^0(\mathcal{O}(mD))^{\otimes \ell} \longrightarrow H^0(\mathcal{O}(\ell m D))$$

$$\Rightarrow \begin{tikzcd} \left(H^0(\mathcal{O}(mD)) \otimes \mathcal{O}(-mD)\right)^{\otimes \ell} \arrow[r] \arrow[dr] & H^0(\mathcal{O}(\ell m D)) \otimes \mathcal{O}(-\ell m D) \arrow[d] \\ & \mathcal{O} \end{tikzcd}$$

Commutes, w/ images the corr. base ideals

$\Rightarrow\ b(|mD|)^{\ell} \subseteq b(|\ell m D|)$

Taking radicals,

$Bs(|\ell m D|) \subseteq Bs(|mD|)$

$\Rightarrow$ X Noetherian $\Rightarrow$ there is some minimal elt.

If $Bs(|pD|)$ and $Bs(|qD|)$ are both minimal, then

$$Bs(|pqD|) = Bs(|pD|) = Bs(|qD|) \qquad \Rightarrow\ \exists\ \underline{\text{unique}}\ \text{min'l elt.}$$

Furthermore, $Bs(|kpD|) = \mathbb{B}(D)\ \forall\ k > 0$. $\square$

<u>Exercise</u>: Come up w/ examples where $m_0 = e(L)$ doesn't work.